CSCI471/971
Modern Cryptography
Zero-Knowledge Proof II

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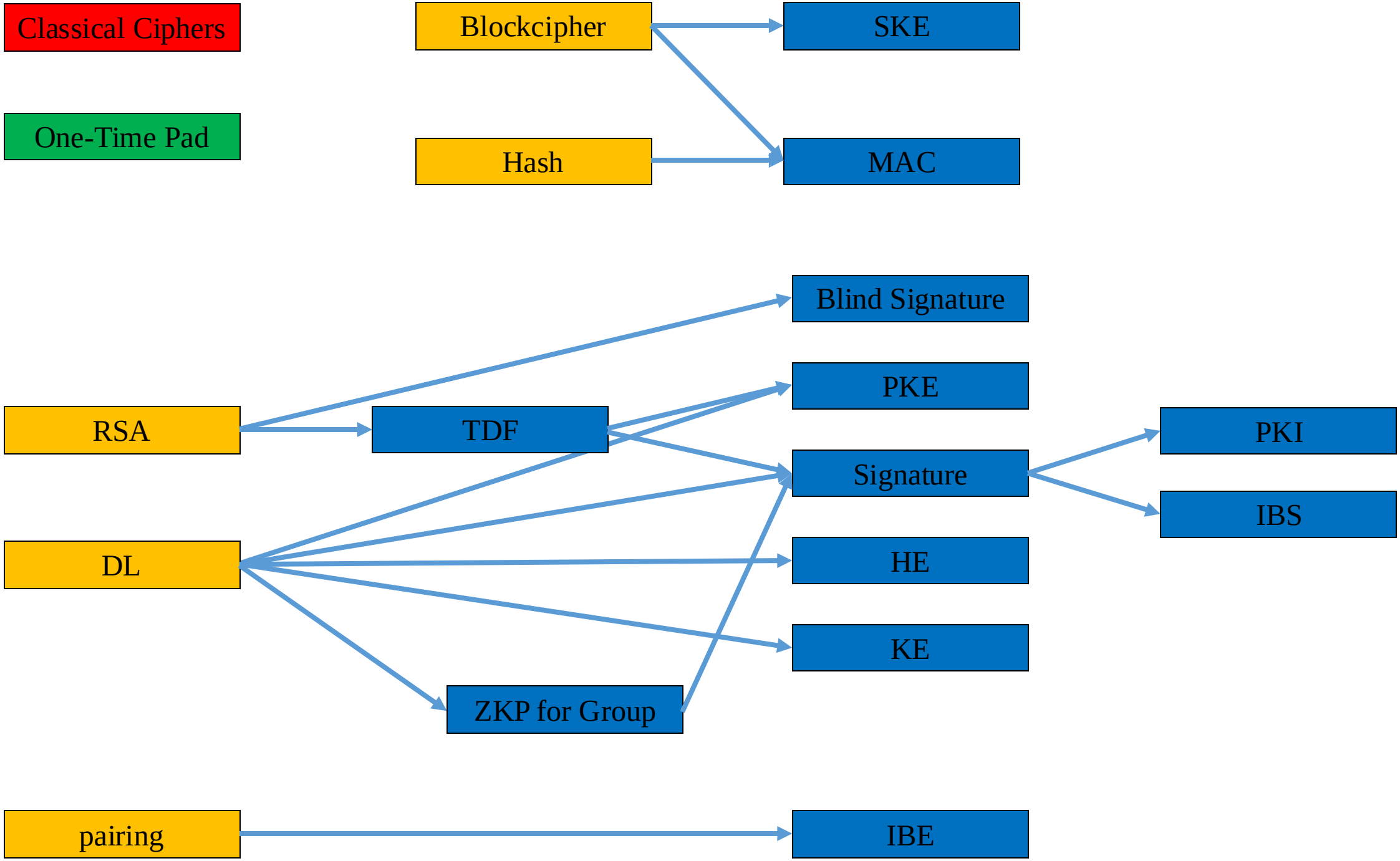
RoadMap

- Week 1-2: Preliminaries
- Week 3-4: Symmetric-Key Cryptography
- Week 5-9: Public-Key Cryptography
- Week 10-11: Zero-Knowledge Proofs


Classical Ciphers
Blockcipher
SKE
One-Time Pad
Hash
MAC
Blind Signature
PKE
RSA
TDF
PKI
Signature
IBS
DL
HE
KE
ZKP for Group
pairing
IBE

Zero-Knowledge Proofs (Revision)

Zero Knowledge Proofs (of knowledge)

- A protocol involving a prover and a verifier.
- The prover takes as input a statement x and a witness w.
- The verifier takes as input the statement x.
- The prover's goal is to convince the verifier that some statement is true (or that she holds the witness w) without revealing any other information.
- Here, we only consider internal attackers, i.e., the prover will try to cheat the verifier and the verifier will try to learn the witness w.

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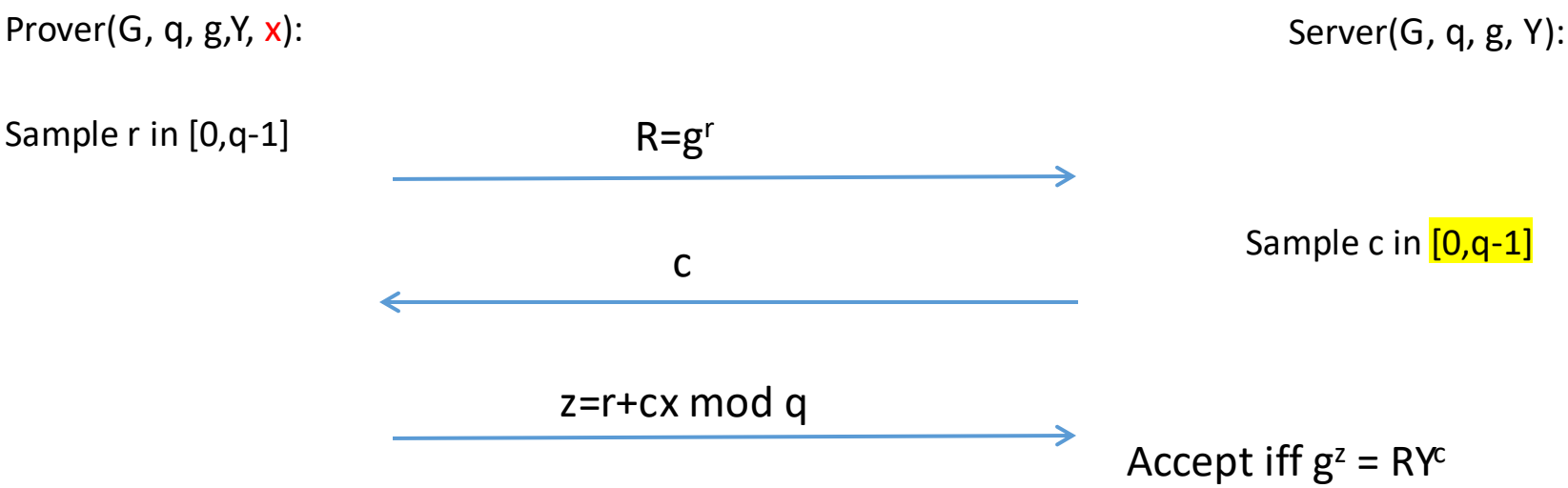
- Correctness:
  - Completeness: Given honest prover and honest verifier, the protocol will output 1
- Security:
  - Soundness: If the statement is wrong (or the prover does not hold the witness), then she cannot pass the verification.
  - Zero-Knowledge: The verifier cannot learn any information from the protocol.
    - Here, we usually cannot use the indistinguishability-based definition (unless each statement is associated with multiple witnesses)
    - We use a simulation-based definition

Preliminaries on Cyclic Group

- Let (G,g,p) be a cyclic group, where G is the set of group element, g is the generator, and p is the group order:
  - $G=\{ g^0, g^1 , \ldots , g^{p-1}\}$
  - $g^p=1$
- The following operations are easy in the group (G,g,p):
  - Given any h_1, h_2 in G, it is easy to compute $h_1 \cdot h_2$
    - For any h in G and for any x,y in [0,p-1], given h^x and h^y, it is easy to compute $h^{x+y}=h^x \cdot h^y$
    - For any h_1, h_2 in G and for any x in[0,p-1], given h_1^x and h_2^x , we can compute $(h_1 \cdot h_2)^x= h_1^x \cdot h_2^x$
  - Given any h in G and any x in [0,p-1], it is easy to compute h^x
- The following operations are hard in the group (G,g,p):
  - Given g^x, it is hard to compute x (The DL problem)
  - Given g^x and g^y, it is hard to compute g^{xy} (The CDH problem)
  - Given g^x and g^y, it is hard to distinguish g^{xy} from a random group element in G (The DDH problem)

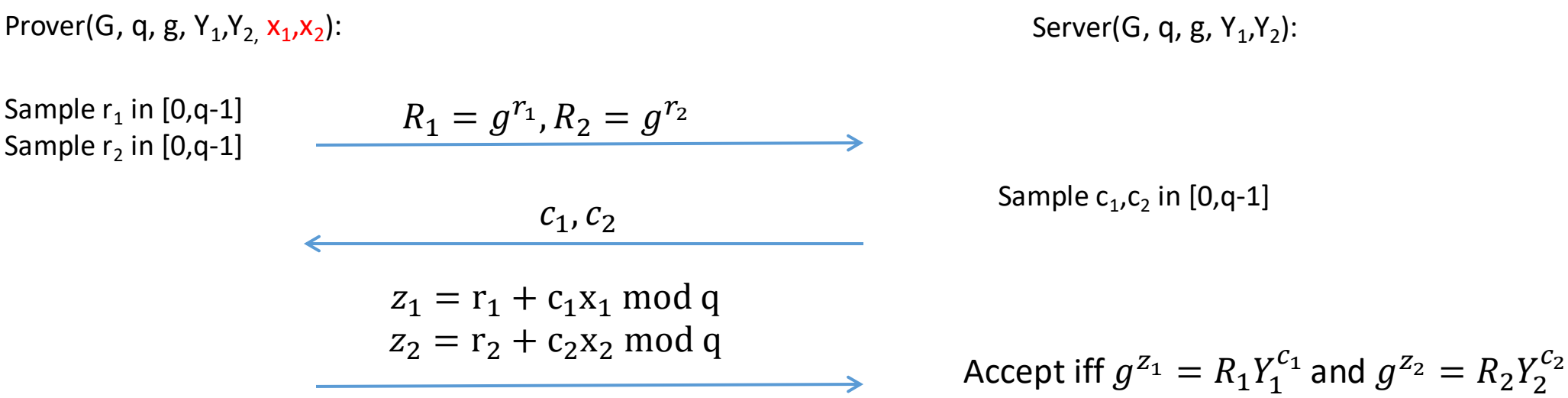
Schnorr Protocol



Zero-Knowledge Proofs for the AND relation in Cyclic Groups

A proof of knowledge of the And Relation

- How to prove that you know the discrete logs x_1, x_2 s.t. $g^{x_1} = Y_1$ and $g^{x_2} = Y_2$?
  - We can simply combine two protocols



A proof of knowledge of the And Relation

- How to prove that you know the discrete log x s.t. $g_1^x = Y_1$ and $g_2^x = Y_2$?
  - Can we combine two protocols directly?

A proof of knowledge of the And Relation

- How to prove that you know the discrete log x s.t. $g_1^x = Y_1$ and $g_2^x = Y_2$?
  - We need to add restrictions to ensure that the two discrete logs are the same.

Prover(G, q, g_1,g_2, Y_1,Y_2, x):

Server(G, q, g_1,g_2, Y_1,Y_2):

Sample r in [0,q-1]

$R_1 = g_1^r, R_2 = g_2^r$ →

Sample c in [0,q-1]

← c

$z = r + cx \bmod q$ →

Accept iff $g_1^z = R_1 Y_1^c$ and $g_2^z = R_2 Y_2^c$

A proof of knowledge of the And Relation

Prover(G, q, g_1,g_2, Y_1,Y_2, x):

Server(G, q, g_1,g_2, Y_1,Y_2):

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$$R_1 = g_1^r, R_2 = g_2^r$$

$$c$$

Sample c in [0,q-1]

$$z = r + cx \bmod q$$

Accept iff $g_1^z = R_1 Y_1^c$ and $g_2^z = R_2 Y_2^c$

- Completeness

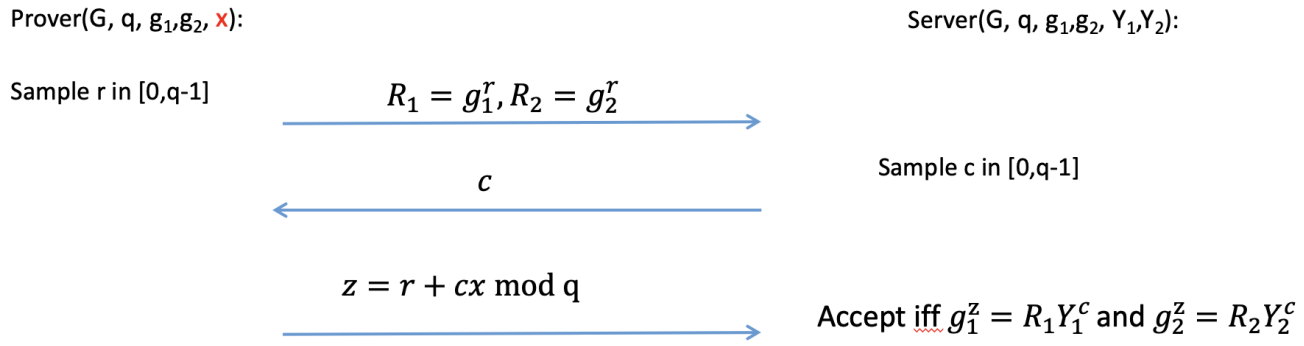
A proof of knowledge of the And Relation

Soundness

- Assume that the prover can always pass the verifications. Then after sending (R_1,R_2), the verifier is able to compute the correct response z on many challenges c.
- Now, based on two different challenges c_1 and c_2, and the correct responses z_1 and z_2, it is easy to extract *x*.
  - Given (R1,R2,c_1,z_1) and (R1,R2,c_2,z_2) where

$$g_1^{z_1} = R1 * Y_1^{c_1} \qquad g_1^{z_2} = R1 * Y_1^{c_2}$$
$$g_2^{z_1} = R2 * Y_2^{c_1} \qquad g_2^{z_2} = R2 * Y_2^{c_2}$$

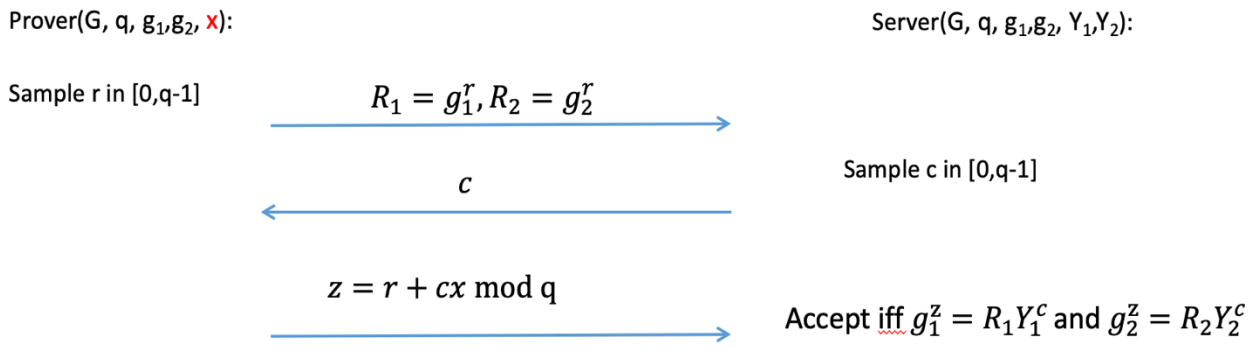
  - We have $g_1^{z_1-z_2} = Y_1^{c_1-c_2}$ and $g_2^{z_1-z_2} = Y_2^{c_1-c_2}$
  - Thus, one can compute x= $(z_1-z_2)(c_1-c_2)^{-1}$ which satisfies $Y_1 = g_1^{x} \wedge Y_2 = g_2^{x}$
- If the prover only passes with a non-negligible probability, a more detailed probabilty analysis and the rewinding techniques are needed.



A proof of knowledge of the And Relation

(Honest-Verifier) Zero-Knowledge

- The verifier only sees a random value
- Actually, one can **simulate the interaction** (R_1,R_2,c,Z) without knowing *x* as long as c is randomly chosen by the prover (i.e., the prover is honest).
  1. Choose a random c from Z_p
  2. Choose a random z
  3. Compute $R_1 = g_1^z * h_1^{-c}, R_2 = g_2^z * h_2^{-c}$(We set r=z-c*x mod q implicitly)



Zero-Knowledge Proofs for the OR relation in Cyclic Groups

A proof of knowledge of the OR Relation

- How to prove that you know the discrete log x s.t. either $g_1^x = Y_1$ or $g_2^x = Y_2$?
  - Assume that you know $g_1^x = Y_1$, then you can at least prove this statement.
  - It implies that you know x s.t. either $g_1^x = Y_1$ or $g_2^x = Y_2$, but it reveals for which part you know the witness.
  - To solve the problem, we need to also include a valid proof for the second part.
    - Fortunately, this is possible if we know the challenge in advance.
  - So, we need to design the protocol in a way that you can know one and only one challenge in advance.

A proof of knowledge of the OR Relation

- How to prove that you know the discrete log x s.t. either $g_1^x = Y_1$ or $g_2^x = Y_2$?

Prover(G, q, g_1,g_2, Y_1,Y_2, x):
W.L.O.G., assume that you know x satisfying $g_1^x = Y_1$

Server(G, q, g_1,g_2, Y_1,Y_2):

Sample r_1, c_2, z_2 in [0,q-1]
$R_1 = g_1^{r_1}, R_2 = g_2^{z_2}/Y_2^{c_2}$

R_1, R_2 →

Sample c in [0,q-1]

← c

$c_1 = c - c_2 \ mod \ q$
$z_1 = r_1 + c_1 x \ mod \ q$

c_1, c_2, z_1, z_2 →

Accept iff
$c = c_1 + c_2 \ mod \ q,$
$g_1^{z_1} = R_1 Y_1^{c_1}$,
$g_2^{z_2} = R_2 Y_2^{c_2}$

A proof of knowledge of the OR Relation

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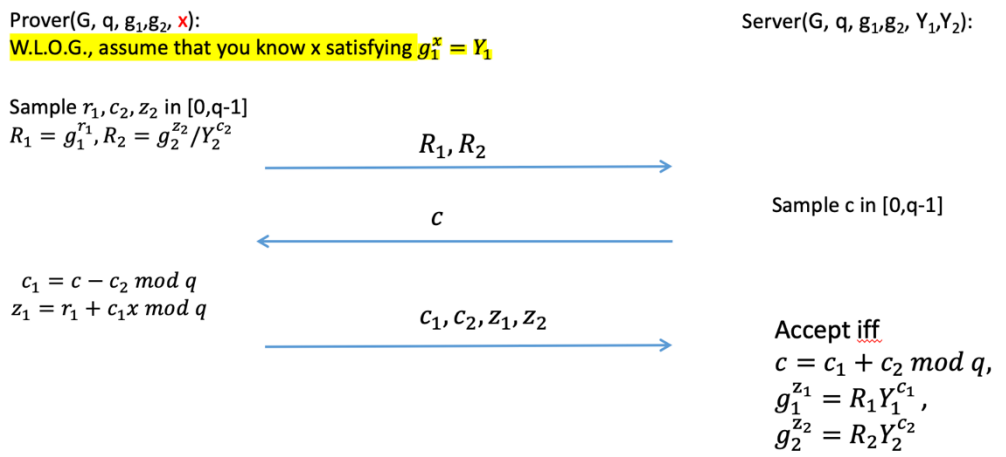
Soundness

- Assume that the prover can always pass the verifications. Then after sending (R_1,R_2), the verifier is able to compute the correct response on many challenges c.
- Now, based on two different challenges c and c', and the correct responses (c_1,c_2,z_1,z_2) and (c'_1,c'_2,z'_1,z'_2) , it is easy to extract *x*.
  - Given (R1,R2,c,c_1,c_2,z_1,z_2) and (R1,R2,c',c'_1,c'_2,z'_1,z'_2) where

$$g_1^{z_1} = R1 * h_1^{c_1} \qquad g_1^{z'_1} = R1 * h_1^{c'_1}$$
$$g_2^{z_2} = R2 * h_2^{c_2} \qquad g_2^{z'_2} = R2 * h_2^{c'_2}$$

  - We have $g_1^{z_1 - z'_1} = h_1^{c_1 - c'_1}$ and $g_2^{z_2 - z'_2} = h_2^{c_2 - c'_2}$
  - As $c \neq c'$, we have either $c_1 \neq c'_1$ or $c_2 \neq c'_2$
  - Thus, one can compute either $x = (z_1 - z'_1)(c_1 - c'_1)^{-1}$ (which satisfies $Y_1 = g_1^x$) or $a = (z_2 - z'_2)(c_2 - c'_2)^{-1}$ (which satisfies $Y_2 = g_2^x$).
- If the prover only passes with a non-negligible probability, a more detailed probabilty analysis and the rewinding techniques are needed.



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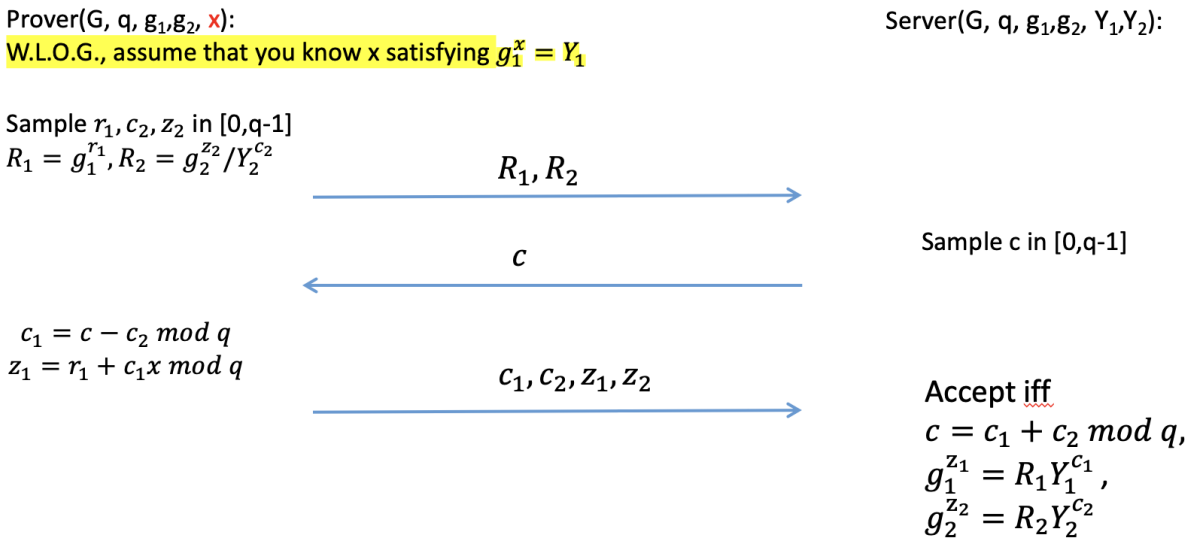
  1. Choose random c_1,c_2, from Z_p
  2. Compute c= c_1+c_2
  3. Choose random z_1,z_2
  4. Compute $R_1 = g_1^{z_1} * h_1^{-c_1}, R_2 = g_2^{z_2} * h_2^{-c_2}$



A proof of knowledge of the OR Relation

- Given $(g_1, Y_1), (g_2, Y_2), \ldots, (g_l, Y_l)$,how to prove that you know one of the discrete log, i.e., a number x s.t. $g_i^x = Y_i$?

Prover(G, q, $(g_1, Y_1), (g_2, Y_2), \ldots, (g_l, Y_l)$, x):
W.L.O.G., assume that you know x satisfying $g_1^x = Y_1$

Server(G, q, $(g_1, Y_1), (g_2, Y_2), \ldots, (g_l, Y_l)$):

Sample r_1 in [0,q-1], $R_1 = g_1^{r_1}$;
For i in [2,l]:
  Sample c_i, z_i in [0,q-1],
  $R_i = g_i^{z_i} / Y_i^{c_i}$

$R_1, R_2, \ldots R_l$ →

← c

Sample c in [0,q-1]

$c_1 = c - \sum_{i=2}^{l} c_i \bmod q$
$z_1 = r_1 + c_1 x \bmod q$

$(c_1, z_1), \ldots, (c_l, z_l)$ →

Accept iff
$c = \sum_{i=1}^{l} c_i \bmod q$,
$g_i^{z_i} = R_i Y_i^{c_i}$ for all i.

Ring Signature

Ring signatures

- In some cases, it is necessary to sign on a message while hiding the identity of the signer.
- Usually, we will require that the public only know that the signature is made by someone in a group, but they do not know the exact identity of the signer.
- For example, in some applications like anonymous reporting, we need to ensure that:
  - The whistleblower can sign the signature on behalf of a set of users (e.g., all staff in a company).
  - Anyone outside this set is not able to sign.
  - No one knows who is the signer/ whistleblower, i.e., the signatures produced by anyone in the set cannot be distinguished.
- The above scenario is a bit artificial, but we finally found some more natural application scenarios. (Assignment 2 Task 1)

Ring Signatures

- KeyGen(λ): Taking as input a security parameter λ, the key generation algorithms returns (pk,sk)

- Sign(sk, M, (pk_1, …, pk_l)): Taking as input a message M, a set of public keys, and a secret key sk for one of the public key, the signing algorithm returns a signature denoted by S.

$$S \leftarrow \text{Sign}(sk, M, (pk_1, \ldots, pk_l))$$

- Verify(S,M, (pk_1, …, pk_l)): Taking as input signed message (S,M) and the set of public keys, the verification algorithm returns 1 or 0.

Ring Signatures

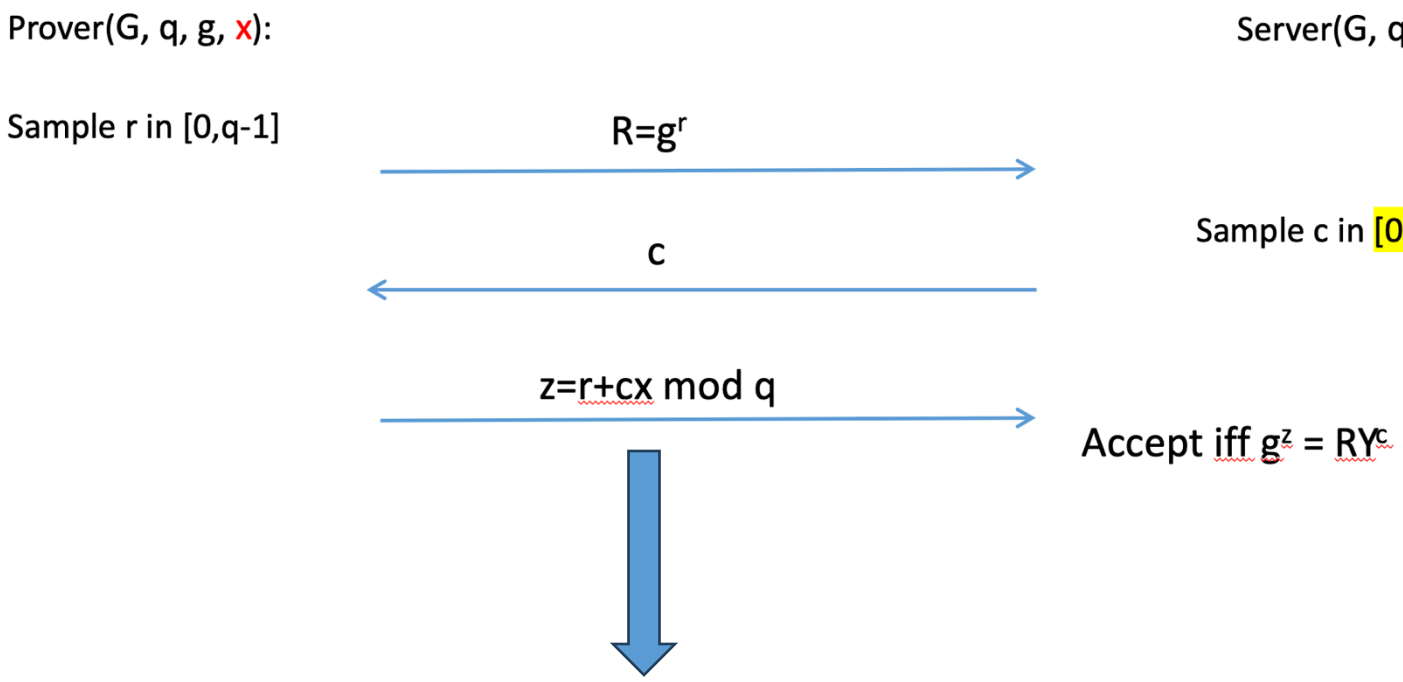
- **<u>Correctness</u>**: For all generated (pk_1,sk_1), …, (pk_l,sk_l), all index i, and all signature $S \leftarrow Sign(sk_i, M, (pk_1, \ldots, pk_l))$, we have

$$Pr[Verify(S,M, (pk_1, \ldots, pk_l))=1]=1$$

Ring Signatures

- Next, let us try to define the security. We need to define unforgeability (since it is a signature) and anonumity of Signer (since it is a ring signature).
- Unforgeability: Anyone outside the set cannot produce a valid signature.
  - The adversary should be able to
    - Aks for a signature on a message M singed by a secret key sk_i on behalf of a set of users.
    - Ask for the secret keys for public keys outside the target set.
  - The goal is to generate a valid signature on a message M* singed by a set R*, where
    - The adversary has not asked for the secret key for any public key in R*.
    - The adversary has not asked the signature for (M*,R*)
- Anonymity: The adversary cannot know the real signer in a group.
  - The adversary is able to know all public keys and secret keys.
    - In some definitions, we require the adversary cannot learn the secret keys of the two targets.
  - The adversary asks for a signature on a message M* and a ring R*, where the signature is singed by either sk_0 or sk_1; the adversary's goal is to distinguish which secret key is used.

Constructing a Ring signature: Warm-Up



In this transform, we transform a proof showing that "I know a secret key of the DL-based cryptosystem" into a signature.

- Sign(sk, M): Taking as input a message M and a secret key sk=(G, q, g, x, H), the P.P.T. algorithm
  1. Choose a random number r and computes $R=g^r$
  2. Compute c=H(R, M)
  3. Compute z=r+ c*x mod q
  4. The signature is (R,z)
- Verify(S,M,pk): Taking as input a signed message M, the public key pk=(G, q, g, h, H), and a signature (R,z), the P.P.T. algorithm
  1. Compute c'=H(R,M) and Accept the signature if $g^z=R\cdot h^{c'}$

Constructing a Ring signature: Warm-Up

Prover(G, q, $(g_1, Y_1), (g_2, Y_2), \dots, (g_l, Y_l)$, x):
W.L.O.G., assume that you know x satisfying $g_1^x = Y_1$

Server(G, q, $(g_1, Y_1), (g_2, Y_2), \dots, (g_l, Y_l)$):

Sample r_1 in [0,q-1], $R_1 = g_1^{r_1}$;
For i in [2,l]:
Sample c_i, z_i in [0,q-1],
$R_i = g_i^{z_i}/Y_i^{c_i}$

$R_1, R_2, \dots R_l$ →

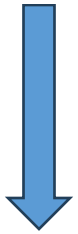
← c

Sample c in [0,q-1]

$c_1 = c - \sum_{i=2}^{l} c_i \bmod q$
$z_1 = r_1 + c_1 x \bmod q$

$(c_1, z_1), \dots, (c_l, z_l)$ →

Accept iff
$c = \sum_{i=1}^{l} c_i \bmod q$,
$g_i^{z_i} = R_i Y_i^{c_i}$ for all i.



The ZKP proves that "I know one secret key (out of l secret keys) of the DL-based cryptosystem". Can we transform it into a ring signature?

Ring Signature

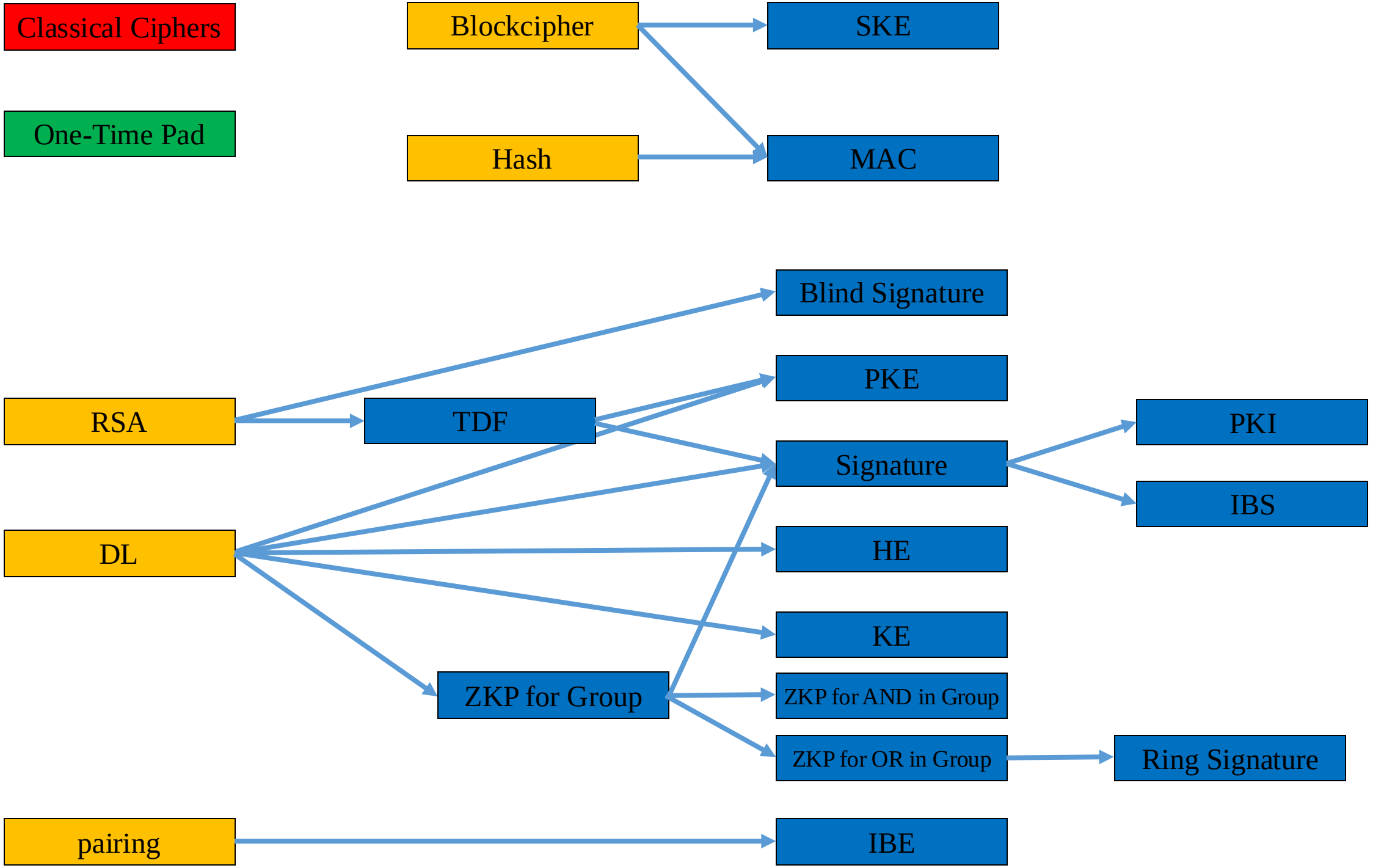
- The signature scheme assume that all parties agree on a cyclic group G of order q, a generator g of G, and a hash function H: {0,1}* $\rightarrow Z_q$

- KeyGen(λ): Taking as input a security parameter λ, the P.P.T. algorithm
  1. Chooses a uniform x ∈ Z_q and compute h = g^x .
  2. The public key is h and the private key is x.

- Sign(sk, M, (h_1, … h_l)): Taking as input a message M, a set of public keys (h_1, … h_l), and a secret key sk=x_i, the P.P.T. algorithm
  1. Choose a random number r_i
  2. Compute $R_i = g^{r_i}$
  3. For j in [1,l] and $j \neq i$:
     1. Choose random c_j, z_j
     2. Compute $R_j = g^{z_j}/h_j^{c_j}$
  4. Compute c=H(R_1, …, R_l, M)
  5. Compute $c_i = c - \sum_{j \neq i} c_j$
  6. Compute $z_i = r_i + c_i * x_i \; mod \; q$
  7. The signature is ((R_1,c_1,z_1), …, (R_l,c_l,z_l))

- Verify(S,M, (h_1, … h_l)): Taking as input a signed message M, a set of public keys (h_1, … h_l), and a signature ((R_1,c_1,z_1), …, (R_l,c_l,z_l)), the P.P.T. algorithm
  1. Compute c'=H(R_1, …, R_l, M)
  2. and Accept the signature if
     1. $c = \sum_{j=1}^{l} c_j \; mod \; q,$
     2. $g^{z_j} = R_j h_j^{c_j}$ for all j

- Correctness.
- Unforgeability
- Anonymity.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ZKP for OR in Group
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pairing
IBE

Summary

- Zero-Knowledge
  - Zero-Knowledge for And Relation
    - Construction
    - Security*
  - Zero-Knowledge for OR Relation
    - General Idea
    - Construction
    - Security*
    - Extending to k statements
  - Ring Signature
    - Definition
      - Application Scenarios
      - Syntax and Correctness
      - Security*
    - Construction from ZKP for OR relations